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Supersymmetric phases of finite-temperature strings II

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Abstract: It was recently proposed that there exist stable supersymmetric phases for finite temperature superstings. This issue was investigated using an effective supergravity which takes into account massive winding modes. Such a theory admits BPS solutions that do not suffer from Hagedorn-type instabilities. We extend several aspects of this work. First we restrict to the real-field sector of the theory and allow, in general, for unequal right and left (ω_+ and ω_-) winding fields. Then, by further specializing to type-II theories (IIA, IIB and a self-dual hybrid) we construct the most general 1/2-BPS solution and reveal several new features arising in various consistent truncations ($\omega_- = \pm\omega_+$, $\omega_- \gg \omega_+$). In the heterotic case we investigate the general properties of the solution which is presented in a closed form in the limit of infinitely large left-winding field ($\omega_- \gg \omega_+$).

Keywords: Superstrings and Heterotic Strings, Supergravity Models, Supersymmetric Effective Theories.

Contents

1. Introduction and generalities

Studying string theory at finite temperature is important due to the relevance for the physics of the very early universe. It is also important for the further understanding of string theory itself, since the interest effects, such as Hagedorn-type instabilities, are due to massive windings string states becoming massless as the temperature is raised. These and related issues were investigated in the past in a series of works, notably in [1]–[8]. In these works, the interactions of massive string states among themselves and with the massless states were either ignored or assumed that they would not changed drastically the physical picture. As far as supersymmetry is concerned, it was always assumed that it necessarily breaks once the temperature is turned on.

The construction of an effective supergravity for finite temperature $N = 4$ superstrings in five-dimensions that takes into account the relevant massive windings states was done for the heterotic and type-II cases in [9] and extended in a U-duality invariant way, that encompasses all perturbative string theories, in [10]. These theories are effectively four dimensional and have as a vacuum solution flat space with zero winding fields and constant moduli, which breaks supersymmetry. Furthermore, the more general theory of [10], reproduces all known Hagedorn-type instabilities for each perturbative string theory separately.

It was only recently that, non-trivial solutions to these theories were found preserving 1/2 of supersymmetries and not suffering from Hagedorn-type instabilities [11]. Their very existence suggests that when superstrings are heated up from zero temperature, they might prefer to go into this more symmetric phase which is stable due to supersymmetry, and that the "ordinary" high-temperature instabilities do not occur, i.e. there is no Hagedorn temperature. It is the purpose of this paper to further pursue this idea and generalize several aspects of the work of [11].

Let us briefly review the most relevant aspects of our effective theory. For details on its construction the reader should consult the original literature [9, 10] and for a shorter review suitable for our purposes and further physical motivation, the ref. [11]. Our effective theory is a four-dimensional $N = 1$ supergravity theory coupled to 9 chiral multiplets corresponding to the 3 complex moduli fields S, T and U and the 6 complex winding fields $Z^{\pm}_A$, $A = 1, 2, 3$. As any generic four-dimensional $N = 1$ supergravity theory it is characterized by a Kähler potential K and a holomorphic superpotential W, but we will not present their explicit expressions here. It was shown in [11] that for any four-dimensional $N = 1$ supergravity theory coupled to chiral multiplets, one may consistently restrict to the real part of the bosonic fields and set the imaginary part to zero. In particular, this procedure leads to 1/2-BPS solutions in a natural way, after examining the Killing spinor equations of the theory. Furthermore, for the theory of [10] describing finite temperature superstrings in a U-duality invariant way, we focused on the real directions defined by $\mathrm{Im}\, Z^{\pm}_A = \mathrm{Im}\, S = \mathrm{Im}\, T = \mathrm{Im}\, U = \mathrm{Re}\,(Z^+_A - Z^-_A) = 0$ which, in total, represent 6 real fields [11]. This consistent truncation lead to important simplifications and made possible to find the general 1/2-BPS solution for the type-II cases and systematically investigate the heterotic case as well.

In this paper we will relax the last of the above conditions that equates the right winding field Z^+_A with the left winding field Z^-_A, and impose only the condition that all fields are real. Hence, we remain with 9 real fields in total which we rename as

$$s = \mathrm{Re}\, S\,, \qquad t = \mathrm{Re}\, T\,, \qquad u = \mathrm{Re}\, U\,, \qquad z^{\pm}_A = \mathrm{Re}\, Z^{\pm}_A\,, \qquad A = 1, 2, 3\,. \tag{1.1}$$

We will see that this generalization reveals some new features of the theory.

Let us introduce for convenience the quantities[1]

$$x^2_{\pm} = \sum_{A=1}^{3} (z^{\pm}_A)^2\,, \qquad H^{\pm}_A = \frac{z^{\pm}_A}{1 - x^2_{\pm}}\,. \tag{1.2}$$

It is also helpful to trade s, t and u for ϕ_1, ϕ_2 and ϕ_3 as follows:

$$s = e^{-2\phi_1}\,, \qquad t = e^{-2\phi_2}\,, \qquad u = e^{-2\phi_3}\,. \tag{1.3}$$

[1] We follow the notation of [11] appropriately modified when necessary.

With these definitions, we obtain a simplified form of the effective supergravity with bosonic lagrangian density

$$\frac{1}{\sqrt{g}}\mathcal{L} = \frac{1}{4}R - \frac{1}{2}\sum_{i=1}^{3}(\partial_\mu\phi_i)^2 - \sum_{A=1}^{3}\frac{(\partial_\mu z_A^+)^2}{(1-x_+^2)^2} - \sum_{A=1}^{3}\frac{(\partial_\mu z_A^-)^2}{(1-x_-^2)^2} - V\,, \tag{1.4}$$

where the scalar potential is given in terms of a superpotential W as

$$V = \frac{1}{4}\sum_{i=1}^{3}\left(\frac{\partial W}{\partial\phi_i}\right)^2 + \frac{1}{8}\sum_{A=1}^{3}(1-x_+^2)^2\left(\frac{\partial W}{\partial z_A^+}\right)^2 + \frac{1}{8}\sum_{A=1}^{3}(1-x_-^2)^2\left(\frac{\partial W}{\partial z_A^-}\right)^2 - \frac{3}{4}W^2\,, \tag{1.5}$$

with

$$\begin{aligned} W &= \frac{1}{2}e^{\phi_1+\phi_2+\phi_3} - 2e^{\phi_1}\sinh(\phi_2+\phi_3)H_1^+H_1^- + e^{-\phi_1}\left(e^{\phi_2-\phi_3}H_2^+H_2^- + e^{\phi_3-\phi_2}H_3^+H_3^-\right) \\ &= \frac{1}{\sqrt{stu}}\left(\frac{1}{2} + (tu-1)H_1^+H_1^- + suH_2^+H_2^- + stH_3^+H_3^-\right). \end{aligned} \tag{1.6}$$

In this paper we will use units where the four-dimensional gravitational coupling κ has been normalized to $\sqrt{2}$.

We make the domain wall ansatz (see, for instance, [12]) for the metric that preserves the symmetries of the three-dimensional space

$$ds^2 = dr^2 + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu\,, \qquad \mu,\ \nu = 1,2,3\,. \tag{1.7}$$

As a result, the domain walls of the theory that preserve 1/2-supersymmetry, correspond to solutions of the non-linear first-order system of differential equations[2]

$$\begin{aligned} \sqrt{2}\frac{d\phi_1}{dr} &= -\frac{1}{2}e^{\phi_1+\phi_+} + 2e^{\phi_1}\sinh\phi_+ H_1^+H_1^- + e^{-\phi_1}\left(e^{\phi_-}H_2^+H_2^- + e^{-\phi_-}H_3^+H_3^-\right), \\ \sqrt{2}\frac{d\phi_2}{dr} &= -\frac{1}{2}e^{\phi_1+\phi_+} + 2e^{\phi_1}\cosh\phi_+ H_1^+H_1^- - e^{-\phi_1}\left(e^{\phi_-}H_2^+H_2^- - e^{-\phi_-}H_3^+H_3^-\right), \\ \sqrt{2}\frac{d\phi_3}{dr} &= -\frac{1}{2}e^{\phi_1+\phi_+} + 2e^{\phi_1}\cosh\phi_+ H_1^+H_1^- + e^{-\phi_1}\left(e^{\phi_-}H_2^+H_2^- - e^{-\phi_-}H_3^+H_3^-\right), \\ \sqrt{2}\frac{dH_1^\pm}{dr} &= e^{\phi_1}\sinh\phi_+\ \left(1+4(H_1^\pm)^2\right)H_1^\mp - \\ &\quad - 2e^{-\phi_1}\left(e^{\phi_-}H_2^+H_2^- + e^{-\phi_-}H_3^+H_3^-\right)H_1^\pm\,, \\ \sqrt{2}\frac{dH_2^\pm}{dr} &= -\frac{1}{2}e^{-\phi_1+\phi_-}H_2^\mp + 4e^{\phi_1}\sinh\phi_+\,H_1^+H_1^-H_2^\pm - \\ &\quad - 2e^{-\phi_1}\left(e^{\phi_-}H_2^+H_2^- + e^{-\phi_-}H_3^+H_3^-\right)H_2^\pm\,, \\ \sqrt{2}\frac{dH_3^\pm}{dr} &= -\frac{1}{2}e^{-\phi_1-\phi_-}H_3^\mp + 4e^{\phi_1}\sinh\phi_+\,H_1^+H_1^-H_3^\pm - \\ &\quad - 2e^{-\phi_1}\left(e^{\phi_-}H_2^+H_2^- + e^{-\phi_-}H_3^+H_3^-\right)H_3^\pm\,, \end{aligned} \tag{1.8}$$

[2]For a detailed explanation of how this system of first order equations is derived from a general theory of scalars coupled to gravity and a potential obtained from a superpotential as in (1.5), see [11, section 3.1]. We also note that, solutions to this system of differential equations, also solve the second order equations corresponding to the lagrangian (1.4).

where we found it more convenient to work with the fields $H_A^{\pm}$ instead of $z_A^{\pm}$ and we defined $\phi_{\pm} = \phi_2 \pm \phi_3$. As soon as a solution has been found, the conformal factor of the metric can be obtained by a simple integration of the resulting expression for the superpotential W, since

$$\frac{dA}{dr} = \frac{1}{\sqrt{2}} W \, . \tag{1.9}$$

This system of equations for 9 real fields is rather complicated and difficult to solve in general. Instead, we will focus attention on subsectors obtained by consistent truncations of the field content. Each one of these consistent truncations results into a system of 4 first-order equations for an equal number of real fields that correspond to the various type II and heterotic theories. It will turn out that the general solutions for all type-II theories can be found explicitly (section 2). For the heterotic case (section 3) a general solution cannot be given in closed form. However, it is possible to extract the general behaviour of the fields in the strong and weak coupling regions as well as around certain critical points. An explicit solution is given in the limit of infinitely large left-winding field. The physical features of our solutions are explained in the appropriate places in the text. In section 4 we summarize our results, we present some directions for future work and also generally comment on various issues related to our approach to superstrings at finite temperature.

2. Type-II string theories

The simplest truncations of the domain wall equations lead to different type-II sectors. We first examine the type-IIA and type-IIB theories, which can be treated simultaneously, and then examine a hybrid type-II sector, which describes type-II strings at the self-dual radius. Both cases turn out to be exactly solvable.

2.1 Type-IIA and -IIB sector

The type-IIA and type-IIB sectors of the theory are obtained by setting

$$\begin{aligned} z_1^{\pm} &= z_2^{\pm} = 0 \, , \qquad \text{for type IIB} \, , \\ z_1^{\pm} &= z_3^{\pm} = 0 \, , \qquad \text{for type IIA} \, , \end{aligned} \tag{2.1}$$

in which case $H_1^{\pm} = H_2^{\pm} = 0$ and $H_1^{\pm} = H_3^{\pm} = 0$, respectively. It follows from (1.8) that ϕ_1 equals ϕ_2 or ϕ_3, respectively, up to an irrelevant additive constant which we will ignore. Then, it is convenient to set

$$\phi_1 = \phi_2 = \frac{\phi}{2} \, , \qquad \phi_3 = \chi \, , \qquad z_3^{\pm} = \tanh\left(\frac{\omega_{\pm}}{2}\right) , \qquad \text{for IIB} \, , \tag{2.2}$$

or

$$\phi_1 = \phi_3 = \frac{\phi}{2} \, , \qquad \phi_2 = \chi \, , \qquad z_2^{\pm} = \tanh\left(\frac{\omega_{\pm}}{2}\right) , \qquad \text{for IIA} \, , \tag{2.3}$$

where we have introduced in either case the fields $\omega^{\pm}$. Then, $2H_2^{\pm} = \sinh\omega_{\pm}$ for the type-IIA theory and similarly $2H_3^{\pm} = \sinh\omega_{\pm}$ for the type-IIB theory. The temperature field in type-II units is $T \sim e^{\phi}$ and the string coupling is $g_s \sim e^{\chi}$.[3] As in [11] we can treat both cases together because the superpotential and the potential assume the same form for the type-IIA and type-IIB theories. Hence, no distinction will be made in the following between the type-IIA or type-IIB theories. The kinetic terms in the lagrangian for the fields χ, ϕ and ω assume the form

$$\mathcal{L}_{\rm kin} = -\frac{1}{2}(\partial\chi)^2 - \frac{1}{4}(\partial\phi)^2 - \frac{1}{4}(\partial\omega_+)^2 - \frac{1}{4}(\partial\omega_-)^2 . \tag{2.4}$$

In that respect the definition for the field ϕ in this paper and in [11] differ by a factor of $\sqrt{2}$.

Explicit calculation shows that in terms of the new variables the superpotential (1.6) becomes

$$W_{\rm II} = \frac{1}{2}e^{\chi}\left(e^{\phi} + \frac{1}{2}e^{-\phi}\sinh\omega_+\sinh\omega_-\right) , \tag{2.5}$$

whereas the corresponding potential (1.5) takes the form

$$V_{\rm II} = \frac{1}{64}e^{2\chi}\left(e^{-2\phi}(\cosh 2\omega_+\cosh 2\omega_- - 1) - 16\sinh\omega_+\sinh\omega_-\right) . \tag{2.6}$$

Using the truncated superpotential $W_{\rm II}$, the type-II domain walls obey the system of first-order equations

$$\begin{aligned}
\frac{d\chi}{dr} &= -\frac{1}{2\sqrt{2}}e^{\chi}\left(e^{\phi} + \frac{1}{2}e^{-\phi}\sinh\omega_+\sinh\omega_-\right) ,\\
\frac{d\phi}{dr} &= -\frac{1}{\sqrt{2}}e^{\chi}\left(e^{\phi} - \frac{1}{2}e^{-\phi}\sinh\omega_+\sinh\omega_-\right) ,\\
\frac{d\omega_\pm}{dr} &= -\frac{1}{2\sqrt{2}}e^{\chi-\phi}\sinh\omega_\mp\cosh\omega_\pm .
\end{aligned} \tag{2.7}$$

Since this system of equations is invariant under $\omega_\pm \to -\omega_\pm$ we will consider the case $\omega_+ \geq 0$ without any loss of generality. Let us also mention that, for $\omega_+ = \omega_-$ all the expressions so far in this section go over to the corresponding expressions in [11, section 4.1].

2.1.1 The general solution

The system of eqs. (2.7) can be completely integrated. First, we easily can show that the two winding fields are related as

$$\begin{aligned}
\omega_+ &= \omega ,\\
\cosh\omega_- &= \lambda\cosh\omega , \qquad \sinh\omega_- = \pm\sqrt{\lambda^2\cosh^2\omega - 1} ,
\end{aligned} \tag{2.8}$$

[3] For explanations on the identifications of the various fields as temperature, string coupling and windings, see [10, 11].

where λ is a positive constant and where, for notational convenience we have introduced the field ω. This relation completely determines ω_- in terms of ω_+ up to an important sign. In fact, the different cases corresponding to the plus and minus signs are associated with different categories of solutions. In particular, when $\omega_+\omega_- < 0$ the potential (2.6) is manifestly positive and there are cannot be any solutions, supersymmetric or not, that lead to tachyonic instabilities. We may restrict to $\lambda \geq 1$ since for $\lambda < 1$ the rôles of ω_+ and ω_- are interchanged.

It is convenient to present the solution for the rest of the fields and the metric by treating the field ω as an independent variable, which is legitimate since the third equation in (2.7) implies that $\omega_+ = \omega$ is a monotonous function of r. Then, the differential equation for ϕ can be easily integrated and also the equation for χ. The resulting family of BPS solutions has

$$e^{-2\phi} = \pm \cosh^2\omega\ \Lambda(\omega)\,, \qquad e^{2\chi} = \cosh^2\omega\ e^{\phi}\,, \tag{2.9}$$

where the $+$ and $-$ signs correspond to the sign of the product $\omega_+\omega_-$. The function $\Lambda(\omega)$ is given by

$$\begin{aligned}\Lambda(\omega) = C + 2\sqrt{y^2-(1+\lambda^2)y+\lambda^2} + (\lambda^2+1)\,\times \\ \times \ln\left(\frac{\lambda^2+1-2y-2\sqrt{y^2-(1+\lambda^2)y+\lambda^2}}{\lambda^2-1}\right)\,, \qquad y = \frac{1}{\cosh^2\omega}\,,\end{aligned}$$

and is parametrized by an arbitrary integration constant C. Another multiplicative integration constant on the r.h.s. of the expression for $e^{2\chi}$ in (2.9) has been omitted, since it can always be absorbed into trivial field redefinitions. The physical interpretation of the solutions depends crucially on a critical value for the constant C. Before we examine this issue let us mention that the equation for the conformal factor of the metric (1.7) is easily integrated and gives $A = -\chi$ (up to a constant that can be absorbed into a redefinition of the x^μ's). Hence, the metric takes the form

$$ds^2 = \frac{8e^{\phi}}{(\lambda^2\cosh^2\omega-1)\cosh^4\omega}\ d\omega^2 + \frac{e^{-\phi}}{\lambda\cosh^2\omega}\ \eta_{\mu\nu}dx^\mu dx^\nu\,. \tag{2.10}$$

The relation between the variables r and ω in (1.7) and (2.10) is given by the relation of differentials in the last equation in (2.7). This integration can be performed to yield $r(\omega)$ in terms of special functions, but it cannot be inverted to get the explicit dependence of $\omega(r)$, except in some limiting cases, as we will see.

In order to find the critical value for the integration constant C let us first note the limiting values for the function $\Lambda(\omega)$, namely

$$\begin{aligned}&\Lambda(0) = C\,, \qquad \Lambda(\infty) = C - C_{\rm crit}\,, \\ &C_{\rm crit} = (\lambda^2+1)\ln\left(\frac{\lambda+1}{\lambda-1}\right) - 2\lambda\ >\ 0\,, \qquad \forall\,\lambda\geq 1\,.\end{aligned} \tag{2.11}$$

It is easily seen that in the interval $[0,\infty)$ the function $\Lambda(\omega)$ is monotonously decreasing between the above two values.

The different cases, as we have seen, correspond to the product $\omega_+\omega_-$ being positive or negative. If any of the winding fields $\omega_\pm$ is taken zero, that implies that the other one is zero as well. The resulting solution in that case was presented in [11] and will not be repeated here.

$\omega_+\omega_- > 0$. Then, the reality condition for the various fields requires that $C > 0$. For $\omega \to 0$ the fields ϕ, χ and ω_- reach some constant values determined by C and the metric becomes the four-dimensional euclidean flat metric.[4] However, whether $\Lambda(\infty)$ is positive, negative or zero distinguishes three different cases: if $C > C_{\rm crit}$ the winding field ω takes values in the entire real line, i.e. $\omega \in [0, \infty)$. In this interval, the temperature field $T \sim e^\phi$ first increases until a maximum value and then it decreases to zero as $\omega \to \infty$. In contrast, the string coupling $g_s \sim e^\chi$ is a monotonously increasing function of ω in the entire interval. If $C = C_{\rm crit}$, then still $\omega \in [0, \infty)$, but now both the temperature and the string coupling are monotonously increasing functions of ω from a constant at $\omega = 0$, to infinity as $\omega \to \infty$. If $C < C_{\rm crit}$, then there exists a maximum value for the winding field, $\omega_{\rm max}$, beyond which the fields become imaginary. As a function of $\omega \in [0, \omega_{\rm max})$ both the temperature T and the string coupling g_s are monotonously increasing from their constant values at $\omega = 0$ to infinity as $\omega \to \omega_{\rm max}^-$.

It is important to work out the expressions for the various fields and the metric near the maximum value of ω, whether that is finite of infinite. In terms of the variable r

$$\begin{aligned}
C > C_{\rm crit}: &\qquad e^{-\omega} \sim r^{2/7}\,, \qquad e^\phi \sim r^{2/7}\,, \qquad e^{2\chi} \sim r^{-2/7}\,, \\
C = C_{\rm crit}: &\qquad e^{-\omega} \sim r^{2/5}\,, \qquad e^\phi \sim r^{-2/5}\,, \qquad e^{2\chi} \sim r^{-6/5}\,, \\
C < C_{\rm crit}: &\qquad \omega_{\rm max} - \omega \sim r^{4/3}\,, \qquad e^\phi \sim e^{2\chi} \sim r^{-2/3}\,,
\end{aligned} \tag{2.12}$$

as $r \to 0^+$. The metric takes the form

$$ds^2 \simeq dr^2 + (\text{const.})\ r^{2\nu}\eta_{\mu\nu}dx^\mu dx^\nu\,, \qquad \text{as } r \to 0^+\,, \tag{2.13}$$

where we have introduced the constant parameter $\nu = 1/7$, $3/5$ and $1/3$ corresponding to C being larger, equal or smaller than $C_{\rm crit}$, respectively. The value at $r = 0$ corresponds to a naked curvature singularity. In order to fully understand the nature of this singularity ultimately we have to go beyond the effective supergravity description. Since, presently this is not feasible, we can study instead whether the propagation of test quantum particles in our backgrounds is well defined. This requires a unique time evolution of initial data or equivalently a unique self-adjoint extension of the wave-operator at the singularity [13, 14, 15]. This criterion has been

[4]This is a universal behaviour and it should be contrasted with the situation in [11] where there was no flat limit when $\omega \to 0$. The reason, as explained below, is that the two solutions are related by a singular limit.

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successfully applied within the AdS/CFT correspondence in many backgrounds related to the Coulomb branch of supersymmetric gauge theories in the sense that the results are in agreement with field theory expectations [16]. This is essentially the reason that we trust this criterion in our case, where there is no independent information on the physical status of our solutions. For more discussion on these issues and an adaptation to metrics with euclidean signature and in particular of the form (2.13), as in our case, see [11, section 6]. It turns out that this criterion requires that for the case of backgrounds that take place in a infinite interval of r, i.e. $r \in [0, \infty)$, the constant $\nu \geq 1/3$. In the cases where the solutions takes place at a finite interval of r with two singularities, one at $r = 0$ and another one at $r = r_0$, the criterion requires that $\nu \geq 1/3$ at one singularity and $\nu < 1/3$ at the other. Returning to our solutions we see that only the family of solutions with $C \leq C_{\text{crit}}$ is "physical".

$\omega_+\omega_- < 0$. Then, if $C < 0$, reality of the fields allows the full range for $\omega \in [0, \infty)$. It turns out that e^{ϕ} ($e^{2\chi}$) decreases (increases) monotonously from a constant value to zero (infinity). The behaviour of the various fields when $\omega \to \infty$ is

$$e^{-\omega} \sim e^{\phi} \sim e^{-2\chi} \sim (r_0 - r)^{2/7}\,, \qquad \text{as } r \to r_0^-\,, \tag{2.14}$$

where r_0 is a positive constant. For the metric we have the form (2.13) with $\nu = 1/7$ and r replaced by $r_0 - r$.

If $0 < C < C_{\text{crit}}$, there exists a minimum value ω_{min} below which the fields become imaginary. In the interval $[\omega_{\text{min}}, \infty)$, e^{ϕ} is a monotonously decreasing function from infinity to zero. Instead, $e^{2\chi}$ between becoming infinity at the two interval-ends, reaches a minimum value. For $\omega \to \omega_{\text{min}}^+$ the various fields behave as

$$\omega - \omega_{\text{min}} \sim r^{4/3}\,, \qquad \epsilon^{\phi} \sim e^{2\chi} \sim r^{-2/3}\,, \qquad \text{as } r \to 0^+\,, \tag{2.15}$$

whereas the metric behaves as in (2.13) with $\nu = 1/3$. For $\omega \to \infty$ the behaviour of the fields and the metric is as in the case with $C < 0$ above. We see that according to our criterion only the family of solutions with $0 < C < C_{\text{crit}}$ is "physical".

Finally, for $C \geq C_{\text{crit}}$ it is impossible to satisfy the reality condition for the fields.

At this point let us note how a disturbing, from a physical view point, feature of our solutions for $0 < C < C_{\text{crit}}$ is resolved to our advantage. For either sign of $\omega_+\omega_-$ the winding field ω is not allowed to take arbitrary positive values, since there is either an upper or a lower bound. As in General Relativity we may try to continue the solution across the singularity occurring at this boundary value of ω. This makes sense also from the point of view that the singularity has been characterized as "physical" according to our criterion. To be concrete, consider the case with $\omega_+\omega_- > 0$ and analytically continue from $\omega < \omega_{\text{max}}$ to $\omega > \omega_{\text{max}}$. Such an analytic continuation brings us to the solution with $\omega_+\omega_- < 0$ with the identification of the two constants $\omega_{\rm max}$ and $\omega_{\rm min}$. This is an elementary mechanism of changing the sign of the winding number in our effective supergravity action approach.[5]

2.1.2 The truncation $\omega_- = \pm\omega_+$

Two cases of particular interest arise when the constant $\lambda = 1$. From (2.8) we have that, either $\omega_- = \omega_+ = \omega$, or $\omega_- = -\omega_+ = -\omega$. In these cases the solution reads

$$e^{-2\phi} = \pm 2\left(\cosh^2\omega(\ln\coth^2\omega + c) - 1\right), \qquad (2.16)$$

where the plus (minus) sign corresponds to equal (opposite) winding fields. This solution can be found by solving the system of eqs. (2.7) after we set $\omega_- = \pm\omega_+ = \pm\omega$, as this is a consistent truncation of the system of equations. It can also be found by taking the limit $\lambda \to 1^+$ in the solution (2.9), after we regularize the resulting infinite expression by also shifting the integration constant C by an infinite positive constant as $C = 2(c-1) - 2\ln(\frac{\lambda-1}{2})$. Since, in this limit $C > 0$ and $C - C_{\rm crit} = 2c$, the solution for $\omega_- = \omega_+$ should be characterized by whether the constant c is positive, zero or negative. Instead, for $\omega_- = -\omega_+$ only $c < 0$ should be allowed. Indeed, the solution for the case $\omega_+ = \omega_-$ was found in [11] with behaviour that depended crucially on whether the constant c was positive, zero or negative in accordance with our general discussion. We will not elaborate in this case, except for the following remark: the solution in [11], with $\omega_- = \omega_+$, does not have as a limiting case for $\omega \to 0$ flat euclidean space with all fields constant. On the other hand we see that this was the limiting behaviour of the general solution (2.9). Then, a paradox seems to arise since (2.16) is a limit of (2.9). The resolution to that comes with the realization, after a closer inspection to the various expressions involved and using the fact that the constant C is shifted by an infinite amount, that the two limits $\omega \to 0$ and $\lambda \to 1^+$ do not commute, since the latter limit is singular.

In the case with $\omega_+ = -\omega_-$ the reality conditions on the fields imposes that $c < 0$ and that there is a minimum value for the winding field, i.e. $\omega \geq \omega_{\rm max}$, in accordance to our general discussion before. In the limit of small or large $|c|$ the constant $\omega_{\rm min}$ can be found analytically

$$\omega_{\rm min} = \begin{cases} -\dfrac{1}{4}\ln\left(\dfrac{-c}{8}\right) & \text{for } c \to 0^- \\ e^{\frac{c-1}{2}} & \text{for } c \to -\infty\,, \end{cases} \qquad (2.17)$$

whereas for intermediate values of c, $\omega_{\rm max}$ ranges between the above two extremes. In this range, $e^{2\phi}$ is a decreasing function of ω. Instead, $e^{2\chi}$ can acquire a minimum

[5]Interestingly, the flipping of the sign of $\omega_+\omega_-$ corresponds to the complexification of the fields as $e^\phi \to ie^\phi$ and $e^\chi \to -ie^\chi$, in such a way that the system of eqs. (2.7) and the solution (2.9) remain real.

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value at some $\omega = \omega_0 > \omega_{\min}$ which, as before, in the limit of large $|c|$, can be computed analytically

$$\omega_0 = \begin{cases} -\frac{1}{4}\ln\left(\frac{-c}{12}\right) & \text{for } c \to 0^- \,, \\ 2^{-1/4}(-c)^{-3/4} & \text{for } c \to -\infty \,, \end{cases} \tag{2.18}$$

whereas for intermediate values of c, ω_0 ranges between the two above extremes. The corresponding value for the field ϕ is given by

$$e^{\phi}|_{\omega=\omega_0} = \frac{1}{\sqrt{2}} \sinh \omega_0 \,. \tag{2.19}$$

The asymptotic behaviour of the various fields near $\omega = \omega_{\min}$ are given by (2.15), whereas for the metric by (2.13) with $\nu = 1/3$. Similarly, for $\omega \to \infty$ the behaviour of the fields is given by (2.14) and for the metric by (2.13) with $\nu = 1/7$, after replacing r by $r_0 - r$.

2.1.3 The truncation in the limit $\omega_- \to \pm\infty$

In is interesting to have a closer look to the case where one of the winding fields becomes much larger than the other one. Without loss of generality we choose $\omega_- \gg \omega_+$. In this case consider the rescaling followed by the limit

$$\phi \longrightarrow \phi + \frac{1}{2}\ln\left(\frac{\lambda}{4}\right) , \qquad \chi \longrightarrow \chi - \frac{1}{2}\ln\left(\frac{\lambda}{4}\right) , \qquad \lambda \to \infty \,, \tag{2.20}$$

and also substitute $\sinh \omega_- \simeq \lambda \cosh \omega$ using (2.8). Then the system of eqs. (2.7) becomes

$$\begin{aligned} \frac{d\chi}{dr} &= -\frac{1}{2\sqrt{2}} e^{\chi} \left(e^{\phi} \pm e^{-\phi} \sinh 2\omega\right) , \\ \frac{d\phi}{dr} &= -\frac{1}{\sqrt{2}} e^{\chi} \left(e^{\phi} \mp e^{-\phi} \sinh 2\omega\right) , \\ \frac{d\omega}{dr} &= \mp\sqrt{2} e^{\chi-\phi} \cosh^2 \omega \,. \end{aligned} \tag{2.21}$$

We emphasize that in this limit the potential (2.6) and the superpotential (2.5) are well defined as one can easily check. In addition, the kinetic term for the field ω in (2.4) is also well defined and becomes $-(1 + \tanh^2 \omega)(\partial\omega)^2/4$. Hence, the limit $\omega_- \to \pm\infty$ can be considered as a consistent truncation of the theory, much like the truncation with $\omega_- = \pm\omega_+$.

The solution of (2.21) in parametric form is

$$\begin{aligned} e^{-2\phi} &= \pm \cosh^2 \omega \left(a - \frac{1}{3} \tanh \omega (3 - \tanh^2 \omega)\right) , \\ e^{2\chi} &= \cosh^2 \omega \; e^{\phi} \,. \end{aligned} \tag{2.22}$$

This solution can be also obtained from (2.9) in the limit $\lambda \to \infty$, after choosing the constant equal to $C = 4a/\lambda$. The different behaviours are found by noticing also that, in this limit $C - C_{\rm crit} = 4(a - 2/3)/\lambda$. Then, for $\omega_- \to +\infty$, we get different behaviours depending on whether $a > 2/3$, $a = 2/3$ or $0 < a < 2/3$. This resembles the behaviour of the general solution for $\omega_+\omega_- > 0$. For $\omega_- \to -\infty$, different behaviours arise depending on whether $a < 0$ or $0 < a < 2/3$ and resemble the behaviour of the general solution for $\omega_+\omega_- < 0$.

2.2 A hybrid type-II sector

There is another truncation of the domain wall equations with $H_1^\pm = 0$, as in type II, which is exactly solvable. We set for this purpose $H_2^\pm = \pm H_3^\pm$ and observe that the full system of equations is consistent provided that $\phi_2 = \phi_3$. As in [11], we call this sector a hybrid of type IIB and type IIA and amounts to choosing the self-dual radius so that there is no distinction between the type-IIA and type-IIB theories.

We proceed further by setting

$$\phi_1 = \chi\,, \qquad \phi_2 = \phi_3 = \frac{1}{2}\phi\,, \qquad z_2^\pm = \pm z_3^\pm = \frac{1}{\sqrt{2}}\tanh\frac{\omega^\pm}{2} \tag{2.23}$$

and so $H_2^\pm = \pm H_3^\pm = \frac{1}{2\sqrt{2}}\sinh\omega_\pm$. The kinetic terms of the fields χ, ϕ and $\omega_\pm$ assumes the same form as in (2.4). As before, the temperature field is $T \sim e^\phi$ and the string coupling is $g_s \sim e^\chi$. Then, the superpotential (1.6) and the corresponding potential become

$$W_{\rm hyb} = \frac{1}{2}\left(e^{\chi+\phi} + \frac{1}{2}e^{-\chi}\sinh\omega_+\sinh\omega_-\right) \tag{2.24}$$

and

$$\begin{aligned} V_{\rm hyb} = {} & \frac{1}{128}e^{-2\chi}\Big(\cosh 2\omega_+\cosh 2\omega_- + \cosh\omega_+ + \cosh\omega_- - 3\Big) - \\ & - \frac{1}{4}e^{\phi}\sinh\omega_+\sinh\omega_-\,, \end{aligned} \tag{2.25}$$

respectively.

The truncated system of equations takes a simpler form in this case, namely

$$\begin{aligned} \frac{d\chi}{dr} &= -\frac{1}{2\sqrt{2}}\left(e^{\chi+\phi} - \frac{1}{2}e^{-\chi}\sinh\omega_+\sinh\omega_-\right), \\ \frac{d\phi}{dr} &= -\frac{1}{\sqrt{2}}e^{\chi+\phi}\,, \\ \frac{d\omega_\pm}{dr} &= -12\sqrt{2}e^{-\chi}\sinh\omega_\mp\cosh\omega_\pm\,. \end{aligned} \tag{2.26}$$

The equation for the conformal factor of the metric (1.7) can be easily integrated; it gives $A = -\chi - \frac{1}{2}\ln(\cosh\omega_+\cosh\omega_-)$, up to a constant that can be absorbed into a redefinition of the x^μ's. Notice that the result is not the same as in the genuine type-II case that we examined before. The system of eqs. (2.26) is invariant under $\omega_\pm \to -\omega_\pm$, and as before we only need to consider the case of $\omega_+ \geq 0$. Similarly to the genuine type-II case, for $\omega_+ = \omega_-$ all the expressions so far in this section go over to the corresponding expressions in [11, section 4.2].

General solution. As before, it is easily seen that the winding fields $\omega_\pm$ are related by (2.8) and similarly we take the constant $\lambda \geq 1$ with no loss of generality. The general solution of the remaining equations gives the following family of BPS solutions with

$$\begin{aligned} e^{-2\phi} &= \pm 4\sqrt{\lambda}\left(F\left(\varphi, \frac{1}{\lambda}\right) - E\left(\varphi, \frac{1}{\lambda}\right) + E\left(\frac{1}{\lambda}\right) - K\left(\frac{1}{\lambda}\right) + B\right), \\ e^{2\chi} &= \frac{e^\phi}{\lambda^{1/2}\cosh\omega}, \qquad \sin\varphi = \frac{1}{\cosh\omega}, \end{aligned} \tag{2.27}$$

where B is a constant of integration and where we have used the standard notation for the complete and incomplete elliptic integrals of the first and the second kind. As before, the $+$ and $-$ signs correspond to the sign of $\omega_+\omega_-$. Also we found that the metric takes the form

$$ds^2 = \frac{8\lambda^{-1/2}e^\phi}{\sqrt{\lambda^2\cosh^2\omega - 1}\,\cosh^3\omega}\,d\omega^2 \;+\; \frac{e^{-\phi}}{\lambda^{1/2}\cosh\omega}\,\eta_{\mu\nu}dx^\mu dx^\nu \,. \tag{2.28}$$

The connection between the variables r and ω in (1.7) and (2.28) is given by a relation of the corresponding differentials. As before, the integration can be performed and yields $r(\omega)$ in a closed form, but it cannot be inverted to get $\omega(r)$, apart from a few limiting cases, as we will see next.

To investigate the properties of the solution we note the limiting values

$$\begin{aligned} e^{-2\phi}|_{\omega=0} &= \pm 4B\,, \qquad e^{-2\phi}|_{\omega=\infty} = \pm 4(B - B_{\text{crit}})\,, \\ B_{\text{crit}} &= K\left(\frac{1}{\lambda}\right) - E\left(\frac{1}{\lambda}\right) \;>\; 0\,, \qquad \forall\,\lambda \geq 1\,. \end{aligned} \tag{2.29}$$

As before, the sign of $\omega_+\omega_-$ distinguishes different types of solutions.

$\omega_+\omega_- > 0$. Then, the reality condition for the various fields requires that $B > 0$. In the limit $\omega \to 0$ the fields ϕ, χ and ω_- approach some constant values determined by B and the metric becomes the four-dimensional euclidean flat metric (see also footnote 4). However, whether B is larger, equal or less than B_{crit} distinguishes three different cases: if $B > B_{\text{crit}}$ then the winding field ω can take values in the entire real line, i.e. $\omega \in [0,\infty)$. In this interval, the temperature field $T \sim e^\phi$ increases monotonously between the two constant values that can be read off from (2.29). The string coupling $g_s \sim e^\chi$ goes from a constant value to zero after reaching a maximum. If $B = B_{\text{crit}}$, then still $\omega \in [0,\infty)$, but now both the temperature and the string coupling are monotonously increasing functions of ω and go from a constant to infinity. If $B < B_{\rm crit}$, then there exists a maximum value for the winding field, $\omega_{\rm max}$, beyond which the fields are not real. As a function of $\omega \in [0, \omega_{\rm max})$ both the temperature T and the string coupling g_s are monotonously increasing from their constant values at $\omega = 0$, to infinity as $\omega \to \infty$.

The behaviour of the various fields and the metric near the maximum value of ω (finite of infinite), in terms of the variable r, is given by

$$\begin{aligned}
&B > B_{\rm crit}: \qquad e^{-\omega} \sim e^{2\chi} \sim r^{2/5}\,, \qquad e^{\phi} \simeq \frac{1}{2}(B - B_{\rm crit})^{-1/2} + ({\rm const.})\; r^{6/5}\,,\\
&B = B_{\rm crit}: \qquad e^{-\omega} \sim r^{4/7}\,, \qquad e^{\phi} \sim r^{-6/7}\,, \qquad e^{2\chi} \sim r^{-2/7}\,,\\
&B < B_{\rm crit}: \qquad \omega_{\rm max} - \omega \sim r^{4/3}\,, \qquad e^{\phi} \sim e^{2\chi} \sim r^{-2/3}\,,
\end{aligned} \tag{2.30}$$

as $r \to 0^+$. The metric takes the form (2.13) with $\nu = 1/5$, $5/7$ and $1/3$ corresponding to B being larger, equal or smaller than $B_{\rm crit}$, respectively. As before, $r = 0$ corresponds to a naked curvature singularity. According to our criterion only the family of solutions with $B \leq B_{\rm crit}$ is "physical".

$\omega_+\omega_- < 0$. Then, if $B < 0$ the reality condition on the fields is obeyed for $\omega \in [0, \infty)$. It turns out that e^{ϕ} and $e^{2\chi}$ decrease monotonously from a constant value to a constant and a zero value, respectively. The behaviour of the various fields when $\omega \to \infty$ is

$$e^{-\omega} \sim e^{2\chi} \sim (r_0 - r)^{2/5}\,, \qquad e^{\phi} \simeq \frac{1}{2}(B_{\rm crit} - B)^{-1/2} + ({\rm const.})\; (r_0 - r)^{6/5}\,, \tag{2.31}$$

where r_0 is a positive constant. The metric we have the form (2.13) with $\nu = 1/5$ and r replaced by $r_0 - r$.

If $0 < B < B_{\rm crit}$, there exists a minimum value $\omega_{\rm min}$ below which the fields are not real. In the interval $[\omega_{\rm min}, \infty)$, e^{ϕ} and $e^{2\chi}$ are monotonously decreasing functions from infinity to a constant and a zero value, respectively. For $\omega \to \omega^+_{\rm min}$ the various fields behave as

$$\omega - \omega_{\rm min} \sim r^{4/3}\,, \qquad \epsilon^{\phi} \sim e^{2\chi} \sim r^{-2/3}\,, \tag{2.32}$$

whereas the metric behaves as in (2.13) with $\nu = 1/3$. For $\omega \to \infty$ the behaviour of the fields and the metric is as in the case with $B < 0$ above. We see that, according to our criterion only the family of solutions with $0 < B < B_{\rm crit}$ is "physical".

For $B \geq B_{\rm crit}$ it is impossible to satisfy the reality condition for the fields.

Let us mention also that the change of the sign of the winding number mechanism, when we analytically continue through the singularity, that was exposed for the genuine type-II case in the paragraph just before section 2.1.2, applies also in this case for $0 < B < B_{\rm crit}$.

2.2.1 The truncation $\omega_+ = \pm\omega_-$

As for the genuine type-II theories, the case when the constant $\lambda = 1$ is of particular interest. Then, from (2.8) we have that either $\omega_- = \omega_+ = \omega$, or $\omega_- = -\omega_+ = -\omega$. In these cases the solution reads

$$e^{-2\phi} = \pm\left(b + 4\left(\ln\left(\coth\frac{\omega}{2}\right) - \frac{1}{\cosh\omega}\right)\right), \tag{2.33}$$

where b is an integration constant. Being a consistent truncation, this solution can be found by solving the system of eqs. (2.26) after we set $\omega_- = \pm\omega_+ = \omega$. It can also be found by taking the limit $\lambda \to 1^+$ in the solution (2.27), after we regularize the resulting infinite expression by also shifting the integration constant B by an infinite positive constant as $B = b/4 - 1 - \frac{1}{2}\ln(\frac{\lambda-1}{8})$. Since in this limit $B > 0$ and $B - B_{\rm crit} = b/4$ the solution for $\omega_- = \omega_+$ should be characterized of whether b is positive, negative or zero, whereas for $\omega_- = -\omega_+$ only $b < 0$ should be allowed. The solution for the case $\omega_+ = \omega_-$ was found in [11] and the behaviour of the solution depended crucially on whether the constant b was positive, zero or negative in accordance with our expectations. Since the limiting procedure that leads to (2.33) from the general solution (2.27) is singular, their corresponding behaviours as $\omega \to 0$ are different.

Turning to the case with $\omega_- = -\omega_+ = -\omega$, we see that the reality condition for the fields χ and ϕ requires that $b < 0$ again in accordance with our expectations and a minimum value for the winding field, i.e. $\omega \geq \omega_{\rm min}$. In this range, $e^{2\chi}$ and e^{ϕ} are monotonically decreasing functions of ω. In the limit of small and large $|b|$ we find for $\omega_{\rm min}$ the analytic expressions

$$\omega_{\rm min} = \begin{cases} -\dfrac{1}{3}\ln\left(\dfrac{-3b}{32}\right) & \text{for } b \to 0^- \\ 2e^{b/4-1} & \text{for } b \to -\infty\,. \end{cases} \tag{2.34}$$

For intermediate values of b, $\omega_{\rm min}$ ranges between the above two extremes.

The asymptotic behaviour of the various fields near $\omega = \omega_{\rm min}$ is given by (2.32), whereas for the metric by (2.13) with $\nu = 1/3$. Similarly, for $\omega \to \infty$ the behaviour of the fields is given by (2.31) and for the metric by (2.13) with $\nu = 1/5$.

2.2.2 The truncation in the limit $\omega_- \to \pm\infty$

Let us consider now the limit of large left-winding field ω_- and in the system of eqs. (2.26) perform the rescaling followed by the limit

$$\phi \longrightarrow \phi + \frac{3}{4}\ln\lambda\,, \qquad \chi \longrightarrow \chi + \frac{1}{8}\ln\lambda\,, \qquad r \longrightarrow r\lambda^{-7/8}\,, \tag{2.35}$$

and also substitute $\sinh\omega_- \simeq \lambda\cosh\omega$ using (2.8). We find the result

$$\begin{aligned}\frac{d\chi}{dr} &= -\frac{1}{2\sqrt{2}}\left(e^{\chi+\phi} \mp \frac{1}{4}e^{-\chi}\sinh 2\omega\right),\\ \frac{d\phi}{dr} &= -\frac{1}{\sqrt{2}}e^{\chi+\phi},\\ \frac{d\omega}{dr} &= \mp\frac{1}{2\sqrt{2}}e^{-\chi}\cosh^2\omega\,,\end{aligned} \tag{2.36}$$

with solution

$$\begin{aligned}e^{-2\phi} &= \pm 2\left(d+\varphi-\frac{\sinh\omega}{\cosh^2\omega}\right),\\ e^{2\chi} &= \frac{e^{\phi}}{\cosh\omega}\,, \qquad \sin\varphi = \frac{1}{\cosh\omega}\,.\end{aligned} \tag{2.37}$$

As before, this limit can be considered as a consistent truncation since the potential, the superspotential and the kinetic term for the field ω are all well defined. Finally, we mention that this solution can be obtained from the general solution in the limit $\lambda \to \infty$ and after we set $B = (d+\pi/2)/(2\lambda^2)$. The different behaviours are found by noticing also that, in this limit $B - B_{\rm crit} = d/(2\lambda^2)$. Then, for $\omega_- \to +\infty$, we get different behaviours depending on whether $d > 0$, $d = 0$ or $-\pi/2 < d < 0$. This resemble the behaviours of the general solution for $\omega_+\omega_- > 0$. For $\omega_- \to -\infty$, different behaviours arise depending on $d < -\pi/2$ or $-\pi/2 < d < 0$ and resemble the behaviour of the general solution for $\omega_+\omega_- < 0$.

3. Heterotic sector

The heterotic limit is obtained by setting $H_2^\pm = H_3^\pm = 0$, while keeping $H_1^\pm$ free to vary. Then, introducing new fields as

$$\phi_1 = \chi\,, \qquad \phi_2 = \phi_3 = \frac{\phi}{2}\,, \qquad z_1^\pm = \tanh\frac{\omega_\pm}{2}\,, \tag{3.1}$$

for which $2H_1^\pm = \sinh\omega_\pm$, we obtain the expression for the truncated superpotential

$$W_{\rm het} = \frac{1}{2}e^{\chi}\left(e^{\phi} - \sinh\phi\sinh\omega_+\sinh\omega_-\right), \tag{3.2}$$

and the potential

$$\begin{aligned}V_{\rm het} = \frac{1}{32}e^{2\chi}\Big(&\cosh 2\phi(\cosh 2\omega_+\cosh 2\omega_- - 1) -\\ &- 2\left(\sinh^2\omega_+ + \sinh^2\omega_- + 4\sinh\omega_+\sinh\omega_-\right)\Big)\,.\end{aligned} \tag{3.3}$$

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The kinetic term for the fields ϕ, χ and $\omega_\pm$ is as in (2.4). The truncated system of differential equations is

$$\begin{aligned}\frac{d\chi}{dr} &= -\frac{1}{2\sqrt{2}}e^{\chi}\left(e^{\phi}-\sinh\phi\sinh\omega_+\sinh\omega_-\right),\\ \frac{d\phi}{dr} &= -\frac{1}{\sqrt{2}}e^{\chi}\left(e^{\phi}-\cosh\phi\sinh\omega_+\sinh\omega_-\right),\\ \frac{d\omega_\pm}{dr} &= \frac{1}{\sqrt{2}}e^{\chi}\sinh\phi\cosh\omega_\pm\sinh\omega_\mp\,.\end{aligned} \tag{3.4}$$

The equation for the conformal factor of the metric (1.7) is easily integrated and gives $A=-\chi$, as in the case of the genuine type-IIA or type-IIB theories. Also, as in the previous cases the winding fields $\omega_\pm$ are related by (2.8) and we may take the constant $\lambda \geq 1$ with no loss of generality. Similar to before, for $\omega_+=\omega_-$ we obtain the corresponding expressions in [11, section 5].

The critical points. These occur in the present case only when $\omega_+\omega_- > 0$ and they are located at

$$\phi=0\,,\qquad \omega_+=\ln\left(\sqrt{1+1/\lambda^2}+1/\lambda\right),\qquad \omega_-=\ln\left(\sqrt{1+\lambda^2}+\lambda\right) \tag{3.5}$$

and at the mirror image point that appears because of the invariance of the equations under $\omega_\pm \to -\omega_\pm$. These points are minima of the potential (3.3) which, when evaluated for either of these points, becomes $V=-\frac{1}{8}e^{2\chi}$ and is independent of the constant λ. The dilaton equation in (3.4) is easily integrated to give at these critical points $e^{-\chi}\sim r$. Recall now that the string frame metric in four dimensions is obtained by (1.7) after multiplying with $e^{2\chi}$. Hence, the resulting background has a flat string metric and a linear dilaton (in the coordinate $\zeta\sim\ln r$).[6] This is an exact solution in string theory. For the case with $\lambda=1$, corresponding to $\omega_+=\omega_-=\ln(\sqrt{2}+1)$ this solution was found, in the presence context, in [9, 10], by solving directly the second order equations of motion following from the action. What we have shown here is that the solution persists for more general values for the windings, parametrized by a constant λ as in (3.5). Finally, we notice that these critical points are saddle points, as a stability analysis around them reveals.

The general solution. The general solution of the system of eqs. (3.4) cannot be found in a closed form. The case with $\omega_-=\omega_+$ was thoroughly investigated in [11] and it has a rather complicated space of solutions (see [11, figure 3]). We expect that the same is true in the more general case with $\omega_+\omega_->0$. In contrast, when $\omega_+\omega_-<0$, it is more or less straightforward to show that a general solution in the

[6] Note that, in this case, it makes sense to pass from the Einstein to the string frame since, when ϕ and ω assume their critical values (3.5), we are left with only massless fields (graviton and dilaton) that couple to a 2-dimensional string world-sheet action.

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ϕ–ω plane starts in the region with $(\phi, \omega) = (-\infty, +\infty)$ and ends in the region with $(\phi, \omega) = (+\infty, +\infty)$. In between, ω acquires a minimum value $\omega_{\min}$ that can be any positive number, depending on the constants of integration. At the two ends we have strong coupling regimes. In particular,

$$e^{-\omega} \sim e^{-\phi} \sim e^{-2\chi} \sim r^{2/7}\,, \qquad \text{as } r \to 0^+ \tag{3.6}$$

and similarly,

$$e^{-\omega} \sim e^{\phi} \sim e^{-2\chi} \sim (r_0 - r)^{2/7}\,, \qquad \text{as } r \to r_0^-\,. \tag{3.7}$$

In both cases the metric takes the form (2.13) with $\nu = 1/7$. According to our criterion this solution is "unphysical".

The truncation in the limit $\omega_- \to \pm\infty$. Consider the rescaling followed by the limit

$$\chi \longrightarrow \chi - \ln\lambda\,, \qquad \lambda \to \infty\,. \tag{3.8}$$

Then, after substituting $\sinh\omega_- \simeq \lambda\cosh\omega$ using (2.8), the system of eqs. (3.4) simplifies to

$$\begin{aligned}
\frac{d\chi}{dr} &= \mp\frac{1}{4\sqrt{2}}e^{\chi}\sinh\phi\sinh 2\omega\,,\\
\frac{d\phi}{dr} &= \mp\frac{1}{2\sqrt{2}}e^{\chi}\cosh\phi\sinh 2\omega\,,\\
\frac{d\omega}{dr} &= \mp\frac{1}{\sqrt{2}}e^{\chi}\sinh\phi\cosh^2\omega\,,
\end{aligned} \tag{3.9}$$

where the upper (lower) sign corresponds to the case with $\omega_+\omega_- > 0$ ($\omega_+\omega_- < 0$). The general solution of this system is given in parametric form by

$$\cosh\phi = (\text{const.})\cosh\omega\,, \qquad e^{2\chi} = \cosh\omega\,, \tag{3.10}$$

where the plus and minus signs are unrelated to the ones in (3.9). It is easily seen that the strong coupling regions exhibits the general behaviour of (3.6) and (3.7).

4. Concluding remarks and further comments

In this paper we pursued further the idea, put forward in [11], that superstrings at finite temperature may reach stable phases and therefore they do not necessarily suffer from Hagedorn-type instabilities. Our investigation was based on an effective supergravity theory that takes into account the relevant winding states of the string and the interactions among themselves and with the massless moduli [9, 10]. First we constructed the most general truncation in the real-field sector of these theories and formulated general conditions for the existence of 1/2-BPS states in this theory. These take the form of 9 coupled, first order non-linear differential equations. In the

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particular cases of the type-IIA, type-IIB and a self-dual hybrid, theories, we presented the most general solutions in closed form and investigated their properties. For the heterotic case the general solution cannot be found in closed form. Nevertheless we analyzed its properties using the system of differential equations that it obeys and presented it in closed form for a particular case.

The solutions that we presented in this paper, compared to the similar ones in [11], allow for unequal left- and right-winding fields. They also have the important feature that flat space with constant moduli is recovered in the limit of small right-winding field. In addition, we have presented a mechanism for changing the sign of the winding field, in an effective action approach, by requiring smooth continuations of our solutions through a curvature singularity.

An issue concerning the supersymmetric solutions found in this paper and also in [11], is whether they suffer from Jens instabilities, typical in thermodynamical systems that contain gravity. A prototype such example was studied in [17] where hot flat space was found to be unstable under long-wavelength density fluctuations of thermal gravitons (expected from the classical Jens instability) as well as due to nucleation of black holes (quantum instability). These instabilities render the thermal canonical ensemble ill defined. In turn, one may wonder if the very foundations of the theories [9, 10] should be doubted in the sense that the identification of the relevant fields $\{S, T, U, Z_A^{\pm}\}$ was based on perturbative string compactifications that were interpreted as thermal ensembles. One is tempted to conclude that such instabilities cannot be avoided in our solutions especially due to the fact that they do not support small volumes and according to general arguments they should collapse into black holes [6]. However, one may present the counter argument that the supersymmetric property of our solutions will ensure their quantum stability as well and that no gravitational collapse will occur. We note that the analysis of [17] was done around hot flat space and supersymmetry was not even an issue. Our spaces have very different asymptotic behaviours and we think that the conclusions of [17] are not directly applicable to our cases.

A perhaps related question is whether an effective action approach can capture a possible breaking of supersymmetry by thermal effects at arbitrarily high mass levels. This is an open question, but it should be noted that higher massive modes can be included in an effective action approach, though technically this is certainly an involved construction. We believe that the inclusion of more states enhances rather than diminishes the chances that supersymmetry can be preserved since it opens up to more possibilities.

Another important issue concerning the supersymmetric solutions found in this paper, as well as those in [11], is to understand their microscopic origin. For that, a "lift" to string or M-theory is necessary. Such a "lift" will also elucidate the origin of the naked singularities that our solutions have. Similar studies have appeared in the recent literature in relation to solutions of five-, four- and seven-dimensional gauged supergravities and the corresponding "lifted" solutions in string or M-theory. Without entering into details we mention that, naked singularities in the lower-dimensional theories sometimes appear benign and have natural interpretations from a higher-dimensional point of view. We expect that at least a subset of our solutions that were characterized as "physical" according to our semi-classical criterion, will survive a real stringy test (see also the remarks after eq. (2.13)). We hope that work along these lines will be reported in the future.

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